Orthogonal polynomials and harmonic analysis

Let $(P_n(x))_{n\in\mathbb{N}_0}$ be an orthogonal polynomial sequence with $P_n(1) \equiv 1$, where orthogonality holds w.r.t. a probability (Borel) measure μ on the real line which satisfies $|\mathsf{supp}\ \mu| = \infty$ and $\mathsf{supp}\ \mu \subseteq (-\infty, 1]$. Moreover, assume that $(P_n(x))_{n\in\mathbb{N}_0}$ satisfies nonnegative linearization of products, i.e., the product of any two $P_m(x), P_n(x)$ is a convex combination of the polynomials $P_{|m-n|}(x), \ldots, P_{m+n}(x)$. Such orthogonal polynomials are accompanied by a Haar measure and a sophisticated harmonic analysis, based on the concept of a polynomial hypergroup which was introduced by Rupert Lasser in the 1980s. In particular, one obtains a certain L^1-algebra (whose properties strongly depend on the underlying sequence $(P_n(x))_{n\in\mathbb{N}_0}$, and which can behave very different from L^1-algebras associated with locally compact groups). Many well-known examples such as Chebyshev polynomials of the first and second kind or Legendre polynomials, which are also important for numerical mathematics, approximation theory and applications, fit into this setting.

Given a concrete sequence $(P_n(x))_{n\in\mathbb{N}_0}$, verifying or disproving nonnegative linearization of products may be very delicate. In the 1970s, George Gasper solved the problem for the Jacobi polynomials $(R_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ by characterizing the parameters $\alpha, \beta > -1$ for which the property holds. In 2005, Ryszard Szwarc, who gave several sufficient general criteria, asked to solve the analogous problem concerning the generalized Chebyshev polynomials $(T_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$; these are the quadratic transformations of the Jacobi polynomials and orthogonal w.r.t. $(1-x^2)^\alpha |x|^{2\beta+1}\,\mathrm{d}x$. In this talk, we present a solution to this problem and show that $(T_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ satisfies the desired property if and only if $(R_n^{(\alpha,\beta)}(x))_{n\in\mathbb{N}_0}$ does.

In the second part of the talk, we consider two further classes: the associated symmetric Pollaczek polynomials are given by recurrence relations of the form (monic normalization) $p_0(x) = 1$, $p_1(x) = x$ and

$$xp_n(x) = p_{n+1}(x) + \frac{(n+\nu)(n+\nu+2\alpha)}{(2n+2\nu+2\alpha+2\lambda+1)(2n+2\nu+2\alpha+2\lambda-1)} p_{n-1}(x)$$

with suitable parameters α, λ, ν. The orthogonalization measure is absolutely continuous, and the Haar measure is of subexponential growth. The little q-Legendre polynomials, however, are orthogonal w.r.t. the purely discrete measure

$$\mu(\{x\}) = \begin{cases} q^n(1-q), & x = 1-q^n \text{ with } n \in \mathbb{N}_0, \\ 0, & \text{else,} \end{cases}$$

$q \in (0,1)$; the Haar measure is of exponential growth. These classes are of particular interest in comparison to the harmonic analysis of locally compact groups, and the talk shall illustrate some differences by presenting results on several amenability properties. Our methods use ingredients like density of idempotents, chain sequences, continued fractions and Turán type inequalities.